Topic 2

Matrices and Systems of Equations

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2.1 Systems of Linear Equations and Their Solutions

2.1.1 Systems of Linear Equations

- A **system of linear equations** is a set of equations that involve the same variables. A **system of linear equations** is a system in which each equation is linear. A **solution** of the system is an assignment of values for the variables that makes *each* equation in the system true.

Example

Linear System

a) $\begin{cases} x - 3y = 1 \\ x + y = 9 \end{cases}$

b) $\begin{cases} 2x - 5y + z = 16 \\ \sqrt{2}x + ey + 4z = 9 \\ x + y = 6 \end{cases}$

Non-linear system

a) $\begin{cases} x^2 - 3y = 1 \\ x + y = 9 \end{cases}$

b) $\begin{cases} 2\sqrt{x} - 5y + z = 16 \\ x + e^y + 4z = 9 \\ x + y = 6 \end{cases}$

2.1 Systems of Linear Equations and Their Solutions

The system of linear equations can be solved by

1) Substitution Method
2) Elimination Method
3) Graphical Method

2.1 Systems of Linear Equations and Their Solutions

2.1.2 Substitution Method

- In the **substitution method** we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

SUBSTITUTION METHOD

1. **Solve for One Variable.** Choose one equation, and solve for one variable in terms of the other variable.
2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

2.1 Systems of Linear Equations and Their Solutions

2.1.3 Elimination Method

- To solve a system using the **elimination method**, we try to combine the equations using sums or differences to eliminate one of the variables.

ELIMINATION METHOD

1. **Adjust the Coefficients.** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.
2. **Add the Equations.** Add the two equations to eliminate one variable, then solve for the remaining variable.
3. **Back-Substitute.** Substitute the value that you found in Step 2 back into one of the original equations, and solve for the remaining variable.

2.1.4 Graphical Method

GRAPHICAL METHOD

1. **Graph Each Equation.** Express each equation in a form suitable for the graphing calculator by solving for y as a function of x. Graph the equations on the same screen.
2. **Find the Intersection Points.** The solutions are the x- and y-coordinates of the points of intersection.

2.1 Systems of Linear Equations and Their Solutions

Example

Given the system of linear equations

$$\begin{cases} 2x + y = 1 & \text{Equation 1} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

Find all solutions of the system using

a) Substitution method

b) Elimination method

c) Graphical method

Solution

a) Substitution Method:

Solve for one variable: We solve for y in the first equation

$$y = 1 - 2x$$

Substitute: Now we substitute for y in the second equation and solve for x

$$\begin{aligned} 3x + 4(1 - 2x) &= 14 \\ 3x + 4 - 8x &= 14 \\ -5x + 4 &= 14 \\ -5x &= 10 \end{aligned}$$

Back-substitute: Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$.

$$y = 1 - 2(-2) = 5$$

Thus $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$.

Solution

b) **Elimination Method**

Multiply equation 1 by -4 we get

$$\begin{cases} -8x - 4y = -4 & \text{Equation 1' (- 4} \times \text{Eq1)} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

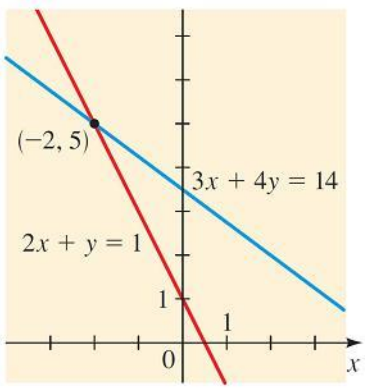
$$\begin{array}{lll} -5x = 10 & & \text{Add} \\ x = -2 & & \text{Solve for } x \end{array}$$

Substitute to equation 1 and solve it

$$2(-2) + y = 1$$
$$y = 5$$

Solution

c) **Graphical method**



2.1 Systems of Linear Equations and Their Solutions

Exercise

Given the system of linear equations

$$\begin{cases} x - 2y = 2 \\ 3x + 2y = 14 \end{cases}$$

Find all solutions of the system using

a) Substitution method

b) Elimination method

c) Graphical method

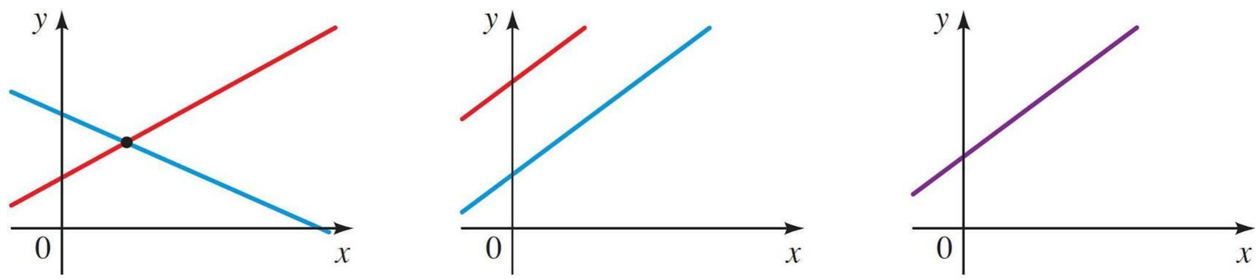
2.1 Systems of Linear Equations and Their Solutions

2.1.5. The Number of Solutions of a Linear System in Two Variables

NUMBER OF SOLUTIONS OF A LINEAR SYSTEM IN TWO VARIABLES

For a system of linear equations in two variables, exactly one of the following is true. (See Figure 5.)

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.



(a) Lines intersect at a single point. The system has one solution.

(b) Lines are parallel and do not intersect. The system has no solution.

(c) Lines coincide—equations are for the same line. The system has infinitely many solutions.

2.1 Systems of Linear Equations and Their Solutions

Example

Solve the following systems:

a) $\begin{cases} 3x - y = 0 \\ 5x + 2y = 22 \end{cases}$

b) $\begin{cases} 8x - 2y = 5 \\ -12x + 3y = 7 \end{cases}$

c) $\begin{cases} 3x - 6y = 12 \\ 4x - 8y = 16 \end{cases}$

Example

Solve the following systems:

a) $\begin{cases} 3x - y = 0 \\ 5x + 2y = 22 \end{cases}$

SOLUTION We eliminate y from the equations and solve for x.

$$\begin{cases} 6x - 2y = 0 \\ 5x + 2y = 22 \end{cases} \qquad \text{2 × Equation 1}$$

$$11x = 22 \qquad \text{Add}$$

$$x = 2 \qquad \text{Solve for } x$$

Now we back-substitute into the first equation and solve for y:

$$6(2) - 2y = 0 \qquad \text{Back-substitute } x = 2$$

$$-2y = -12 \qquad \text{Subtract } 6 \times 2 = 12$$

$$y = 6 \qquad \text{Solve for } y$$

The solution of the system is the ordered pair (2, 6), that is,

$$x = 2, \qquad y = 6$$

The graph in Figure 6 shows that the lines in the system intersect at the point (2, 6).

Example

Solve the following systems:

b) $\begin{cases} 8x-2y=5 \\ -12x+3y=7 \end{cases}$

Solution:

This time we try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second equation by 2 gives

$$\begin{cases} 24x - 6y = 15 & 3 \times \text{Equation 1} \\ -24x + 6y = 14 & 2 \times \text{Equation 2} \end{cases}$$

$$0 = 29 \quad \text{Add}$$

0= 29 is false. Thus the system has no solution.

Example

Solve the following systems:

c) $\begin{cases} 3x-6y=12 \\ 4x-8y=16 \end{cases}$

Solution:

We multiply the first equation by 4 and the second by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 & 4 \times \text{Equation 1} \\ 12x - 24y = 48 & 3 \times \text{Equation 2} \end{cases}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line give a solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$

$$y = \tfrac{1}{2}t - 2$$

We can also write the solution in ordered-pair form as

$$\left(t, \tfrac{1}{2}t - 2\right)$$

where t is any real number. The system has infinitely many solutions (see Figure 8).

Exercise

Solve the following systems:

a) $\begin{cases} 3x + 4y = 10 \\ x - 4y = -2 \end{cases}$

b) $\begin{cases} 2x - 6y = 6 \\ -x + 3y = -3 \end{cases}$

c) $\begin{cases} 2x - 3y = 5 \\ 10x - 15y = 2 \end{cases}$

2.2 Matrices and Systems of Linear Equations

2.2.1 Matrices

DEFINITION OF MATRIX

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

m rows

n columns

We say that the matrix has **dimension** $m \times n$. The numbers a_{ij} are the **entries** of the matrix. The subscript on the entry a_{ij} indicates that it is in the ith row and the jth column.

2.2.1 Matrices

Example

Matrix	Dimension
$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 4 & 9 \end{bmatrix}$	2 x 3 (2 rows by 3 columns)
$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$	1x4 (1 row by 4 columns)

2.2 Matrices and Systems of Linear Equations

2.2.2 Augmented Matrix of a Linear System

- Matrix representation of linear system is denoted as augmented matrix which consists of coefficients and constants of the system.

Example

Linear System

$$\begin{cases} 3x-2y+z=5 \\ x+3y-z=0 \\ -x+\quad 4z=11 \end{cases}$$

Augmented Matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

2.2 Matrices and Systems of Linear Equations

Exercise

Write the augmented matrix of the following system of linear equations

System	Augmented matrix
$\begin{cases} x + y - z = 1 \\ x + 2z = -3 \\ 2y - z = 3 \end{cases}$	$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$

2.2.3 The Elementary Row Operations

ELEMENTARY ROW OPERATIONS

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Note that performing any of these operations on the augmented matrix of a system does not change its solution

Notation

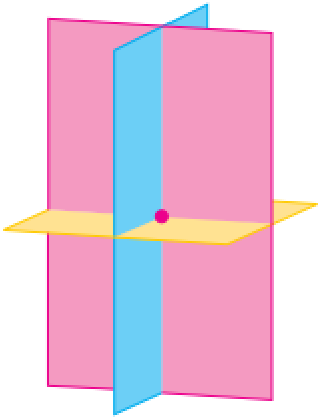
Symbol	Description
$R_i + kR_j \rightarrow R_i$	Change the *i*th row by adding *k* times row *j* to it, and then put the result back in row *i*.
kR_i	Multiply the *i*th row by *k*.
$R_i \leftrightarrow R_j$	Interchange the *i*th and *j*th rows.

2.2 MATRICES AND SYSTEMS OF LINEAR EQUATIONS

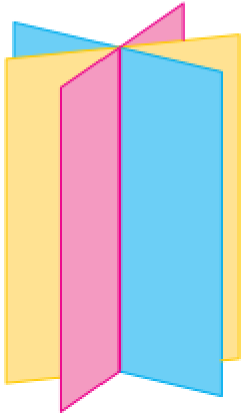
Intersection of Three Planes

A linear equation in three variables is a plane in a three-dimensional coordinate system. For a system of three equations in three variables, the following situation arise:

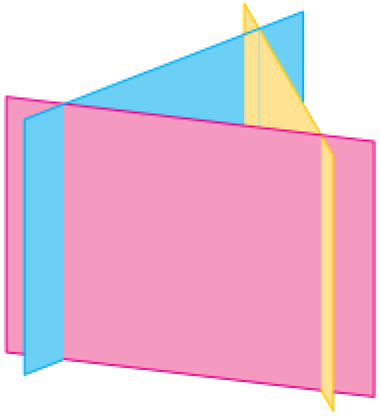
1. The three planes intersect in a single point.
   The system has a unique solution.



2. The three planes intersect in more than one point.
   The system has infinitely many solutions.



3. The three planes have no point in common.
   The system has no solution.



Example

Using elementary row operations to solve a linear system $\begin{cases} x - y + 3x = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$

Solution:

	System		Augmented matrix
	$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$		$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
Add $(-1) \times$ Equation 1 to Equation 2. Add $(-3) \times$ Equation 1 to Equation 3.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$
Multiply Equation 3 by $\frac{1}{2}$.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{\frac{1}{2}R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Add $(-3) \times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).	$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{R_2 - 3R_3 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Interchange Equations 2 and 3.	$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases}$	$\xrightarrow{R_2 \leftrightarrow R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

By back substitution we get   and . Thus the solution is (2, 7, 3).

2.2.4 Gaussian Elimination

- In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

2.2 Matrices and Systems of Linear Equations

ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM OF A MATRIX

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do *not* shift to the right in successive rows

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them

2.2 Matrices and Systems of Linear Equations

Way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Example

Solve the system of linear equations, using Gaussian elimination with back-substitution

$$\begin{cases} 4x+8y-4z=4 \\ 3x+8y+5z=-11 \\ -2x+y+12z=-17 \end{cases}$$

2.2 Matrices and Systems of Linear Equations

Solution:

Need a 1 here

Augmented matrix:

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Need 0's here

$$\xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Need a 1 here

$$\xrightarrow[R_3 + 2R_1 \rightarrow R_3]{R_2 - 3R_1 \rightarrow R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

Need a 1 here

$$\xrightarrow{R_3 - 5R_2 \rightarrow R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix}$$

Row-echelon form:

$$\xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution:

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ y + 4z = -7 \\ z = -2 \end{cases}$$

Back-substitute: We use back-substitution to solve the system.

$y + 2(-2) = -3$ Back-substitute $z = -2$ into Equation 2

$y = 1$ Solve for y

$x + 2(1) - (-2) = 1$ Back-substitute $y = 1$ and $z = -2$ into Equation 1

$x = -3$ Solve for x

So the solution of the system is $(-3, 1, -2)$.

2.2.5 Gauss-Jordan Elimination

If we put the augmented matrix of a linear system in *reduced* row-echelon form, then we don't need to back substitute to solve the system.

To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Example

Solve the system of linear equations, using Gauss-Jordan elimination

$$\begin{cases} 4x+8y-4z=4 \\ 3x+8y+5z=-11 \\ -2x+y+12z=-17 \end{cases}$$

Solution

Need 0's here

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow[R_1 + R_3 \to R_1]{R_2 - 4R_3 \to R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Hence we immediately arrive at the solution $(-3, 1, -2)$.

Exercise

The augmented matrix of a system of linear equations is given in reduced row-echelon form. Find the solution of the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(a)	(b)	(c)
$x =$ _____	$x =$ _____	$x =$ _____
$y =$ _____	$y =$ _____	$y =$ _____
$z =$ _____	$z =$ _____	$z =$ _____

Exercise

Given a system of linear equation

$$\begin{cases} x-2y+z=1 \\ \quad y+2z=5 \\ x+y+3z=8 \end{cases}$$

Find the solution using

a) Gaussian elimination

b) Gauss-Jordan elimination

2.2.6 Inconsistent and Dependent Systems

THE SOLUTIONS OF A LINEAR SYSTEM IN ROW-ECHELON FORM

Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the row-echelon form contains a row that represents the equation $0 = c$, where c is not zero, then the system has no solution. A system with no solution is called **inconsistent**.
2. **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.
3. **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables and if the system is not inconsistent, then it has infinitely many solutions. In this case the system is called **dependent**. We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable

2.2 Matrices and Systems of Linear Equations

Example

Solve the system $\begin{cases} x-3y+2z=12 \\ 2x-5y+5z=14 \\ x-2y+3z=20 \end{cases}$

Solution: We transform the system into row-echelon form

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{bmatrix} \xrightarrow{\frac{1}{18}R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By translate the last row back into equation form we get

$$0x+0y+0z=1$$

or

$$0=1$$

Which is false. Hence the system has **no solution**.

Example

Find the complete solution of the system $\begin{cases} -3x - 5y + 36z = 10 \\ -x \qquad\quad + 7z = 5 \\ \;\; x + \;\; y - 10z = -4 \end{cases}$

Solution: We transform the system into reduce row-echelon form

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} x \qquad - 7z = -5 & \text{Equation 1} \\ \quad y - 3z = \;\; 1 & \text{Equation 2} \end{cases}$$

Leading variables

Now solve the leading variables x and y in terms of non-leading variable z:

$x = 7z - 5$ Solve for x in Equation 1

$y = 3z + 1$ Solve for y in Equation 2

For general solution let $z = t$ we get

$$x = 7t - 5$$
$$y = 3t + 1$$
$$z = t$$

Thus the general solution as the ordered triple $(7t-5, 3t+1, t)$ where t denotes the real number.

Exercise

Solve the system of linear equations

a) $$\begin{cases} x+y+z=2 \\ y-3x=1 \\ 2x+y+5z=0 \end{cases}$$

b) $$\begin{cases} x-2y+5z=3 \\ -2x+6y-9x=-3 \\ 3x-16y+10z=-6 \end{cases}$$

c) $$\begin{cases} 3r+2s+3t=10 \\ r-s-t=-5 \\ 5r-5t=0 \end{cases}$$

2.3 The Algebra of Matrices

2.3.1 Equality of Matrices

EQUALITY OF MATRICES

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same dimension $m \times n$, and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example

Find a, b, c and d if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

Solution:

Since the two matrices are equal, corresponding entries must be the same.

Thus

$a = 1$, $b = 3$, $c = -2$ and $d = 5$

2.3 The Algebra of Matrices

2.3.2 Addition, Subtraction, and Scalar Multiplication of Matrices

SUM, DIFFERENCE, AND SCALAR PRODUCT OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.

$$A - B = [a_{ij} - b_{ij}]$$

3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.

$$cA = [ca_{ij}]$$

2.3.2 Addition, Subtraction, and Scalar Multiplication of Matrices

Example Let

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

Carry out each indicated operation, or explain why it cannot be performed.

(a) $A + B$ **(b)** $C - D$ **(c)** $C + A$ **(d)** $5A$

Solution:

(a) $A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$

(b) $C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$

(c) $C + A$ is undefined because we can't add matrices of different dimensions.

(d) $5A = 5 \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$

2.3 The Algebra of Matrices

2.3.2 Addition, Subtraction, and Scalar Multiplication of Matrices

PROPERTIES OF ADDITION AND SCALAR MULTIPLICATION OF MATRICES

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = cdA$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

2.3 The Algebra of Matrices

2.3.3 Matrix Multiplication

MATRIX MULTIPLICATION

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix

$$C = [c_{ij}]$$

where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as

$$C = AB$$

2.3.3 Matrix Multiplication

Example

Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$

Calculate if possible AB and BA

2.3.3 Matrix Multiplication

SOLUTION Since A has dimension 2×2 and B has dimension 2×3, the product AB is defined and has dimension 2×3. We can therefore write

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1 \cdot (-1) + 3 \cdot 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows.

Entry	Inner product of:	Value	Product matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 5 + 3 \cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 2 + 3 \cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot (-1) + 0 \cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 5 + 0 \cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 2 + 0 \cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

2.3 THE ALGEBRA OF MATRICES

2.3.3 Matrix Multiplication

Thus, we have $$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \qquad \text{and} \qquad 2 \times 2$$

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product.

PROPERTIES OF MATRIX MULTIPLICATION

Let A, B, and C be matrices for which the following products are defined. Then

$$A(BC) = (AB)C \qquad \text{Associative Property}$$

$$A(B + C) = AB + AC$$
$$(B + C)A = BA + CA \qquad \text{Distributive Property}$$

NOTE: In general matrix multiplication is not commutative i.e. .

2.3.4 Transpose of a Matrix

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging rows and columns of A i.e.

Example

$$\left(A^T\right)_{ij} = A_{ji}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2.3.4 Transpose of a Matrix

Example

Let $A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$

Calculate the products AB and BA.

SOLUTION Since both matrices A and B have dimension 2×2, both products AB and BA are defined, and each product is also a 2×2 matrix.

$$AB = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 7 \cdot 9 & 5 \cdot 2 + 7 \cdot (-1) \\ (-3) \cdot 1 + 0 \cdot 9 & (-3) \cdot 2 + 0 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot (-3) & 1 \cdot 7 + 2 \cdot 0 \\ 9 \cdot 5 + (-1) \cdot (-3) & 9 \cdot 7 + (-1) \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix}$$

This shows that, in general, $AB \neq BA$. In fact, in this example AB and BA don't even have an entry in common.

2.3.4 Transpose of a Matrix

Example

$$A=\begin{bmatrix}2 & 5\\ -1 & 7\end{bmatrix}\qquad B=\begin{bmatrix}0 & -1 & 3\\ 2 & 3 & 1\end{bmatrix}\qquad C=\begin{bmatrix}1\\ -2\\ 7\end{bmatrix}\qquad D=\begin{bmatrix}3 & 4\\ 2 & 0\end{bmatrix}$$

Perform the matrix operation or if it is impossible explain why

i) $3A$ ii) $A-2D$ iii) $B-C$ iv) AD

v) DA vi) AB vii) B^TA viii) BC

2.3.5 Inverse of Matrix

2.3.5.1 Identity Matrix

IDENTITY MATRIX

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad I_n \cdot B = B$$

whenever these products are defined.

2.3.5 Inverse of Matrix

2.3.5.2 Inverse of 2x2 Matrix

INVERSE OF A MATRIX

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A.

INVERSE OF A 2 × 2 MATRIX

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A has no inverse.

2.3.5 Inverse of Matrix

2.3.5.2 Inverse of 2x2 Matrix

Example

Let

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Find A^{-1}, and verify that $AA^{-1} = A^{-1}A = I_2$.

SOLUTION Using the rule for the inverse of a 2×2 matrix, we get

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A, we calculate AA^{-1} and $A^{-1}A$:

$$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4(-\frac{5}{2}) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2(-\frac{5}{2}) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \cdot 4 + (-\frac{5}{2})2 & \frac{3}{2} \cdot 5 + (-\frac{5}{2})3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.3.5 Inverse of Matrix

2.3.5.3 Inverting n x n Matrices using Elementary Row Operations

Let A be a square matrix of order n. The elementary row operations can be used to compute since $AA^{-1} = I = A^{-1}A$. If we augment A by I and transform A into I on the augmented matrix by using the elementary row operations (i.e. Gauss-Jordan Elimination), then I on the augmented matrix will be transformed into A^{-1} . If the process to transform A into I is not possible, then it is said that A is not invertible.

2.3.5.3 Inverting an *n* x *n* Matrices using the Adjoint Matrix

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Where the adjoint matrix is defined as transpose of the cofactor matrix

$$\operatorname{adj}(A) = \left[A_{ij} \right]^T$$

and A_{ij} is the cofactor of element a_{ij}.

2.3 The Algebra of Matrices

2.3.6 Determinant of Matrix

2.3.6.1 Determinant of a 2x2 matrix

DETERMINANT OF A 2 × 2 MATRIX

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example:

Evaluate $|A|$ for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$.

SOLUTION

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$

2.3.6 Determinant of Matrix

2.3.6.2 Determinant of an nxn Matrix

The determinant of a matrix obtained using cofactors of the elements in the first row of the matrix is known as **expanding the determinant by first row**.

THE DETERMINANT OF A SQUARE MATRIX

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

2.3.6 Determinant of Matrix

Example

Evaluate the determinant of the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$

Solution:

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2 \cdot 6 - 4 \cdot 5) - 3[0 \cdot 6 - 4(-2)] - [0 \cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

Note that determinant of a matrix can be obtained via expanding the determinant by any row or column

Example

From the example above compute the determinant of A by expanding the determinant

a) by the second row

b) by the third column

Solution

(a) Expanding by the second row, we get

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 0 + 2[2 \cdot 6 - (-1)(-2)] - 4[2 \cdot 5 - 3(-2)]$$

$$= 0 + 20 - 64 = -44$$

(b) Expanding by the third column gives

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= -[0 \cdot 5 - 2(-2)] - 4[2 \cdot 5 - 3(-2)] + 6(2 \cdot 2 - 3 \cdot 0)$$

$$= -4 - 64 + 24 = -44$$

2.3.6 Determinant of Matrix

2.3.6.2.1 Minors and Cofactors

MINORS AND COFACTORS

Let A be an $n \times n$ matrix.

1. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.
2. The **cofactor** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Example:

Let $A=\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$

Find

i) The minors M_{12} and M_{33}

ii) Then find the cofactors A_{12} and A_{33}

Solution

i)

$$M_{12} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

$$M_{33} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$

ii) $A_{12} = (-1)^{1+2} \times M_{12} = -1 \times 8 = -8$

$A_{33} = (-1)^{3+3} \times M_{33} = 1 \times 4 = 4$

Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Thus in a 3×3 matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example :

Find the inverse of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Solution

The determinant of A is $\det(A) = 1\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 6 & -3 \end{vmatrix} = 1$

Matrix of Cofactor

$$\left[A_{i\,j}\right] = \begin{bmatrix} -2 & -3 & -2 \\ -3 & -3 & -4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{transpose}} Adj(A) = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Therefore, the inverse of A is $A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{1}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$

2.3.7 Solving Linear System in form of Matrix Equation

SOLVING A MATRIX EQUATION

If A is a square $n \times n$ matrix that has an inverse A^{-1} and if X is a variable matrix and B a known matrix, both with n rows, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

References

- Stewart, J., Redlin, L., & Watson, S. (2012) Precalculus: Mathematics for Calculus (6th ed,). Brooks/Cole, Cengage Learning.